

FVN Documentation

William Daniau

February 15, 2008

Contents

1	Whatis fvn,licence,disclaimer etc	3
1.1	Whatis fvn	3
1.2	Licence	3
1.3	Disclaimer	4
2	Naming scheme and convention	4
3	Linear algebra	4
3.1	Matrix inversion	4
3.2	Matrix determinants	5
3.3	Matrix condition	5
3.4	Eigenvalues/Eigenvectors	6
3.5	Sparse solving	7
3.6	Identity matrix	8
4	Interpolation	9
4.1	Quadratic Interpolation	9
4.1.1	One variable function	9
4.1.2	Two variables function	10
4.1.3	Three variables function	11
4.1.4	Utility procedure	12
4.2	Akima spline	12
4.2.1	Interpolation	12
4.2.2	Evaluation	12
4.2.3	Example	13
5	Least square polynomial	14
6	Zero finding	15
7	Numerical integration	17
7.1	Gauss Legendre Abscissas and Weigth	17
7.2	Gauss Legendre Numerical Integration	18
7.3	Gauss Kronrod Adaptative Integration	18
7.3.1	Numerical integration of a one variable function	18
7.3.2	Numerical integration of a two variable function	19

8	Special functions	20
8.1	Elementary functions	21
8.1.1	carg	21
8.1.2	cbrt	21
8.1.3	exprl	21
8.1.4	log10	21
8.1.5	alnrel	21
8.2	Trigonometry	21
8.2.1	tan	21
8.2.2	cot	22
8.2.3	sindg	22
8.2.4	cosdg	22
8.2.5	asin	22
8.2.6	acos	22
8.2.7	atan	22
8.2.8	atan2	23
8.2.9	sinh	23
8.2.10	cosh	23
8.2.11	tanh	23
8.2.12	asinh	23
8.2.13	acosh	23
8.2.14	atanh	24
8.3	Exponential Integral and related	24
8.3.1	ei	24
8.3.2	e1	24
8.3.3	ali	24
8.3.4	si	24
8.3.5	ci	25
8.3.6	cin	25
8.3.7	shi	25
8.3.8	chi	25
8.3.9	cinh	25
8.4	Gamma function and related	26
8.4.1	fac	26
8.4.2	binom	26
8.4.3	gamma	26
8.4.4	gamr	26
8.4.5	alngam	26
8.4.6	algams	27
8.4.7	gami	27
8.4.8	gamic	27
8.4.9	gamit	27
8.4.10	psi	28
8.4.11	poch	28
8.4.12	poch1	28
8.4.13	beta	28
8.4.14	albeta	29
8.4.15	betai	29
8.5	Error function and related	29
8.5.1	erf	29
8.5.2	erfc	29
8.5.3	daws	29
8.6	Bessel functions and related	30
8.6.1	bsj0	30

8.6.2	bsj1	30
8.6.3	bsjn	30
8.6.4	bsy0	31
8.6.5	bsy1	31
8.6.6	bsyn	31
8.6.7	bsi0	32
8.6.8	bsi1	32
8.6.9	bsin	33
8.6.10	bsk0	33
8.6.11	bsk1	33
8.6.12	bskn	34
8.6.13	bsi0e	34
8.6.14	bsi1e	35
8.6.15	bsk0e	35
8.6.16	bsk1e	35
8.6.17	bsks	36
8.6.18	bskes	36
8.7	Airy function and related	36
8.7.1	ai	36
8.7.2	bi	36
8.7.3	aid	37
8.7.4	bid	37
8.7.5	aie	37
8.7.6	bie	37
8.7.7	aide	37
8.7.8	bide	38
8.8	Miscellaneous functions	38
8.8.1	spenc	38
8.8.2	inits	38
8.8.3	csevl	38

1 Whatis fvn,licence,disclaimer etc

1.1 Whatis fvn

fvn is a Fortran95 mathematical module. It provides various usefull subroutine covering linear algebra, numerical integration, least square polynomial, spline interpolation, zero finding, special functions etc.

Most of the work is done by interfacing Lapack <http://www.netlib.org/lapack> which means that Lapack and Blas <http://www.netlib.org/blas> must be available on your system for linking fvn. If you use an AMD microprocessor, the good idea is to use ACML (AMD Core Math Library <http://developer.amd.com/acml.jsp> which contains an optimized Blas/Lapack. Fvn also contains a slightly modified version of Quadpack <http://www.netlib.org/quadpack> for performing the numerical integration tasks. Finally the fnlib library <http://www.netlib.org/fn> has been added for special functions.

This module has been initially written for the use of the “Acoustic and microsonic” group led by Sylvain Ballandras in the Time and Frequency Department of institute Femto-ST <http://www.femto-st.fr/>.

1.2 Licence

The licence of fvn is free. You can do whatever you want with this code as far as you credit the authors.

Authors

As of the day this manual is written there's only one author of fvn :
William Daniau
william.daniau@femto-st.fr

1.3 Disclaimer

The usual disclaimer applied : This software is provided AS IS in the hope it will be usefull. Use it at your own risks. The authors should not be taken responsible of anything that may result by the use of this software.

2 Naming scheme and convention

The naming scheme of the routines is as follow :

`fvn*_name()`

where * can be s,d,c or z.

- s is for single precision real (real,real*4,real(4),real(kind=4))
- d for double precision real (double precision,real*8,real(8),real(kind=8))
- c for single precision complex (complex,complex*8,complex(4),complex(kind=4))
- z for double precision complex (double complex,complex*16,complex(8),complex(kind=8))

In the following description of subroutines parameters, input parameters are followed by (in), output parameters by (out) and parameters which are used as input and modified by the subroutine are followed by (inout).

For each routine, there is a generic interface (simply remove `_*` in the name), so using the specific routine is not mandatory.

3 Linear algebra

The linear algebra routines of fvn are an interface to lapack, which make it easier to use.

3.1 Matrix inversion

call `fvn_matinv(d,a,inva,status)`

- d (in) is an integer equal to the matrix rank
- a (in) is a real or complex matrix. It will remain untouched.
- inva (out) is a real or complex matrix which contain the inverse of a at the end of the routine
- status (out) is an optional integer equal to zero if something went wrong

Example

```
program inv
  use fvn
  implicit none

  real(8),dimension(3,3) :: a,inva

  call random_number(a)
  a=a*100

  call fvn_matinv(3,a,inva)
  write (*,*) a
  write (*,*)
  write (*,*) inva
  write (*,*)
  write (*,*) matmul(a,inva)
end program
```

3.2 Matrix determinants

```
det=fvn_det(d,a,status)
```

- d (in) is an integer equal to the matrix rank
- a (in) is a real or complex matrix. It will remain untouched.
- status (out) is an optional integer equal to zero if something went wrong

Example

```
program det
  use fvn
  implicit none

  real(8),dimension(3,3) :: a
  real(8) :: deta
  integer :: status

  call random_number(a)
  a=a*100

  deta=fvn_det(3,a,status)
  write (*,*) a
  write (*,*)
  write (*,*) "Det = ",deta
end program
```

3.3 Matrix condition

```
call fvn_matcon(d,a,rcond,status)
```

- d (in) is an integer equal to the matrix rank
- a (in) is a real or complex matrix. It will remain untouched.

- `rcond` (out) is a real of same kind as matrix `a`, it will contain the reciprocal condition number of the matrix
- `status` (out) is an optional integer equal to zero if something went wrong

The reciprocal condition number is evaluated using the 1-norm and is define as in equation 1

$$R = \frac{1}{\text{norm}(A) * \text{norm}(\text{inv}A)} \quad (1)$$

The 1-norm itself is defined as the maximum value of the columns absolute values (modulus for complex) sum as in equation 2

$$L1 = \max_j \left(\sum_i |A(i,j)| \right) \quad (2)$$

Example

```

program cond
  use fvn
  implicit none

  real(8),dimension(3,3) :: a
  real(8) :: rcond
  integer :: status

  call random_number(a)
  a=a*100

  call fvn_d_matcon(3,a,rcond,status)
  write (*,*) a
  write (*,*)
  write (*,*) "Cond = ",rcond
end program

```

3.4 Eigenvalues/Eigenvectors

```
call fvn_matev(d,a,evala,eveca,status)
```

- `d` (in) is an integer equal to the matrix rank
- `a` (in) is a real or complex matrix. It will remain untouched.
- `evala` (out) is a complex array of same kind as `a`. It contains the eigenvalues of matrix `a`
- `eveca` (out) is a complex matrix of same kind as `a`. Its columns are the eigenvectors of matrix `a`: `eveca(:,j)=jth` eigenvector associated with eigenvalue `evala(j)`.
- `status` (out) is an optional integer equal to zero if something went wrong

Example

```

program eigen
  use fvn
  implicit none

  real(8),dimension(3,3) :: a

```

```

complex(8),dimension(3) :: evala
complex(8),dimension(3,3) :: eveca
integer :: status,i,j

call random_number(a)
a=a*100

call fvn_matev(3,a,evala,eveca,status)
write (*,*) a
write (*,*)
do i=1,3
  write(*,*) "Eigenvalue ",i,evala(i)
  write(*,*) "Associated Eigenvector :"
  do j=1,3
    write(*,*) eveca(j,i)
  end do
  write(*,*)
end do

end program

```

3.5 Sparse solving

By interfacing Tim Davis's SuiteSparse from university of Florida <http://www.cise.ufl.edu/research/sparse/SuiteSparse/> which is a reference for this kind of problems, fvn provides simple subroutines for solving linear sparse systems.

The provided routines solves the equation $Ax = B$ where A is sparse and given in its triplet form.

```
call fvn_sparse_solve(n,nz,T,Ti,Tj,B,x,status)
```

- For this family of subroutine the two letters (zl,zi,dl,di) of the specific interface name describe the arguments's type. z is for complex(8), d for real(8), l for integer(8) and i for integer(4)
- n (in) is an integer equal to the matrix rank
- nz (in) is an integer equal to the number of non-zero elements
- T(nz) (in) is a complex/real array containing the non-zero elements
- Ti(nz),Tj(nz) (in) are the indexes of the corresponding element of T in the original matrix.
- B(n) (in) is a complex/real array containing the second member of the equation.
- x(n) (out) is a complex/real array containing the solution
- status (out) is an integer which contain non-zero is something went wrong

Example

```

program test_sparse

use fvn
implicit none

integer(8), parameter :: nz=12

```

```

integer(8), parameter :: n=5
complex(8),dimension(nz) :: A
integer(8),dimension(nz) :: Ti,Tj
complex(8),dimension(n) :: B,x
integer(8) :: status

A = (/ (2.,0.),(3.,0.),(3.,0.),(-1.,0.),(4.,0.),(4.,0.),(-3.,0.),&
      (1.,0.),(2.,0.),(2.,0.),(6.,0.),(1.,0.) /)
B = (/ (8.,0.),(45.,0.),(-3.,0.),(3.,0.),(19.,0.)/)
Ti = (/ 1,2,1,3,5,2,3,4,5,3,2,5 /)
Tj = (/ 1,1,2,2,2,3,3,3,3,4,5,5 /)

!specific routine that will be used here
!call fvn_zl_sparse_solve(n,nz,A,Ti,Tj,B,x,status)
call fvn_sparse_solve(n,nz,A,Ti,Tj,B,x,status)
write(*,*) x

end program

program test_sparse

use fvn
implicit none

integer(4), parameter :: nz=12
integer(4), parameter :: n=5
real(8),dimension(nz) :: A
integer(4),dimension(nz) :: Ti,Tj
real(8),dimension(n) :: B,x
integer(4) :: status

A = (/ 2.,3.,3.,-1.,4.,4.,-3.,1.,2.,2.,6.,1. /)
B = (/ 8., 45., -3., 3., 19./)
Ti = (/ 1,2,1,3,5,2,3,4,5,3,2,5 /)
Tj = (/ 1,1,2,2,2,3,3,3,3,4,5,5 /)

!specific routine that will be used here
!call fvn_di_sparse_solve(n,nz,A,Ti,Tj,B,x,status)
call fvn_sparse_solve(n,nz,A,Ti,Tj,B,x,status)
write(*,*) x

end program

```

3.6 Identity matrix

```
I=fvn*_ident(n)      (*=s,d,c,z)
```

- n (in) is an integer equal to the matrix rank

This function return the identity matrix of rank n, in the specified type. No generic interface for this one.

4 Interpolation

4.1 Quadratic Interpolation

fvn provide function for interpolating values of a tabulated function of 1, 2 or 3 variables, for both single and double precision.

4.1.1 One variable function

```
value=fvn_quad_interpol(x,n,xdata,ydata)
```

- x is the real where we want to evaluate the function
- n is the number of tabulated values
- xdata(n) contains the tabulated coordinates
- ydata(n) contains the tabulated function values $ydata(i)=y(xdata(i))$

xdata must be strictly increasingly ordered. x must be within the range of xdata to actually perform an interpolation, otherwise the resulting value is an extrapolation

Example

```
program inter1d

use fvn
implicit none

integer(kind=4),parameter :: ndata=33
integer(kind=4) :: i,nout
real(kind=8) :: f,fdata(ndata),h,pi,q,sin,x,xdata(ndata)
real(kind=8) :: tv

intrinsic sin

f(x)=sin(x)

xdata(1)=0.
fdata(1)=f(xdata(1))
h=1./32.
do i=2,ndata
    xdata(i)=xdata(i-1)+h
    fdata(i)=f(xdata(i))
end do
call random_seed()
call random_number(x)

q=fvn_d_quad_interpol(x,ndata,xdata,fdata)

tv=f(x)
write(*,*) "x ",x
write(*,*) "Calculated (real) value :",tv
write(*,*) "fvn interpolation :",q
write(*,*) "Relative fvn error :",abs((q-tv)/tv)

end program
```

4.1.2 Two variables function

```
value=fvn_quad_2d_interpol(x,y,nx,xdata,ny,ydata,zdata)
```

- x,y are the real coordinates where we want to evaluate the function
- nx is the number of tabulated values along x axis
- xdata(nx) contains the tabulated x
- ny is the number of tabulated values along y axis
- ydata(ny) contains the tabulated y
- zdata(nx,ny) contains the tabulated function values $zdata(i,j)=z(xdata(i),ydata(j))$

xdata and ydata must be strictly increasingly ordered. (x,y) must be within the range of xdata and ydata to actually perform an interpolation, otherwise the resulting value is an extrapolation

Example

```
program inter2d
use fvn
implicit none

integer(kind=4),parameter  :: nx=21,ny=42
integer(kind=4)  :: i,j
real(kind=8)  :: f,fdata(nx,ny),dble,pi,q,sin,x,xdata(nx),y,ydata(ny)
real(kind=8)  :: tv

intrinsic dble,sin

f(x,y)=sin(x+2.*y)
do i=1,nx
    xdata(i)=dble(i-1)/dble(nx-1)
end do
do i=1,ny
    ydata(i)=dble(i-1)/dble(ny-1)
end do
do i=1,nx
    do j=1,ny
        fdata(i,j)=f(xdata(i),ydata(j))
    end do
end do
call random_seed()
call random_number(x)
call random_number(y)

q=fvn_d_quad_2d_interpol(x,y,nx,xdata,ny,ydata,fdata)
tv=f(x,y)

write(*,*) "x y",x,y
write(*,*) "Calculated (real) value :",tv
write(*,*) "fvn interpolation :",q
write(*,*) "Relative fvn error :",abs((q-tv)/tv)

end program
```

4.1.3 Three variables function

```
value=fvn_quad_3d_interpol(x,y,z,nx,xdata,ny,ydata,nz,zdata,tdata)
```

- x,y,z are the real coordinates where we want to evaluate the function
- nx is the number of tabulated values along x axis
- xdata(nx) contains the tabulated x
- ny is the number of tabulated values along y axis
- ydata(ny) contains the tabulated y
- nz is the number of tabulated values along z axis
- zdata(nz) contains the tabulated z
- tdata(nx,ny,nz) contains the tabulated function values $tdata(i,j,k)=t(xdata(i),ydata(j),zdata(k))$

xdata, ydata and zdata must be strictly increasingly ordered. (x,y,z) must be within the range of xdata and ydata to actually perform an interpolation, otherwise the resulting value is an extrapolation

Example

```
program inter3d
use fvn

implicit none

integer(kind=4),parameter  :: nx=21,ny=42,nz=18
integer(kind=4)  :: i,j,k
real(kind=8)  :: f,fdata(nx,ny,nz),dble,pi,q,sin,x,xdata(nx),y,ydata(ny),z,zdata(nz)
real(kind=8)  :: tv

intrinsic dble,sin

f(x,y,z)=sin(x+2.*y+3.*z)
do i=1,nx
    xdata(i)=2.*(dble(i-1)/dble(nx-1))
end do
do i=1,ny
    ydata(i)=2.*(dble(i-1)/dble(ny-1))
end do
do i=1,nz
    zdata(i)=2.*(dble(i-1)/dble(nz-1))
end do
do i=1,nx
    do j=1,ny
        do k=1,nz
            fdata(i,j,k)=f(xdata(i),ydata(j),zdata(k))
        end do
    end do
end do
end do
call random_seed()
call random_number(x)
```

```

call random_number(y)
call random_number(z)

q=fvn_d_quad_3d_interpol(x,y,z,nx,xdata,ny,ydata,nz,zdata,fdata)
tv=f(x,y,z)

write(*,*) "x y z",x,y,z
write(*,*) "Calculated (real) value :",tv
write(*,*) "fvn interpolation :",q
write(*,*) "Relative fvn error :",abs((q-tv)/tv)

end program

```

4.1.4 Utility procedure

fvn provides a simple utility procedure to locate the interval in which a value is located in an increasingly ordered array.

```
call fvn_find_interval(x,i,xdata,n)
```

- x (in) the real value to locate
- i (out) the resulting indice
- xdata(n) (in) increasingly ordered array
- n (in) size of the array

The resulting integer i is as : $xdata(i) \leq x < xdata(i+1)$. If $x < xdata(1)$ then $i = 0$ is returned. If $x > xdata(n)$ then $i = n$ is returned. Finally if $x = xdata(n)$ then $i = n - 1$ is returned.

4.2 Akima spline

fvn provides Akima spline interpolation and evaluation for both single and double precision real.

4.2.1 Interpolation

```
call fvn_akima(n,x,y,br,co)
```

- n (in) is an integer equal to the number of points
- x(n) (in) ,y(n) (in) are the known couples of coordinates
- br (out) on output contains a copy of x
- co(4,n) (out) is a real matrix containing the 4 coefficients of the Akima interpolation spline for a given interval.

4.2.2 Evaluation

```
y=fvn_spline_eval(x,n,br,co)
```

- x (in) is the point where we want to evaluate
- n (in) is the number of known points and br(n) (in), co(4,n) (in) are the outputs of fvn_x_akima(n,x,y,br,co)

4.2.3 Example

In the following example we will use Akima splines to interpolate a sinus function with 30 points between -10 and 10. We then use the evaluation function to calculate the coordinates of 1000 points between -11 and 11, and write a 3 columns file containing : x, calculated $\sin(x)$, interpolation evaluation of $\sin(x)$.

One can see that the interpolation is very efficient even with only 30 points. Of course as soon as we leave the -10 to 10 interval, the values are extrapolated and thus can lead to very inaccurate values.

```
program akima
  use fvn
  implicit none

  integer :: nbpoints,nppoints,i
  real(8),dimension(:),allocatable :: x_d,y_d,breakpoints_d
  real(8),dimension(:,:),allocatable :: coeff_fvn_d
  real(8) :: xstep_d,yp_d,ty_d,fvn_y_d

  open(2,file='fvn_akima_double.dat')
  open(3,file='fvn_akima_breakpoints_double.dat')
  nbpoints=30
  allocate(x_d(nbpoints))
  allocate(y_d(nbpoints))
  allocate(breakpoints_d(nbpoints))
  allocate(coeff_fvn_d(4,nbpoints))

  xstep_d=20./dfloat(nbpoints)
  do i=1,nbpoints
    x_d(i)=-10.+dfloat(i)*xstep_d
    y_d(i)=dsin(x_d(i))
    write(3,44) (x_d(i),y_d(i))
  end do
  close(3)

  call fvn_d_akima(nbpoints,x_d,y_d,breakpoints_d,coeff_fvn_d)

  nppoints=1000
  xstep_d=22./dfloat(nppoints)
  do i=1,nppoints
    xp_d=-11.+dfloat(i)*xstep_d
    ty_d=dsin(xp_d)
    fvn_y_d=fvn_d_spline_eval(xp_d,nbpoints-1,breakpoints_d,coeff_fvn_d)
    write(2,44) (xp_d,ty_d,fvn_y_d)
  end do

  close(2)

44      FORMAT(4(1X,1PE22.14))

end program
```

Results are plotted on figure [1](#)

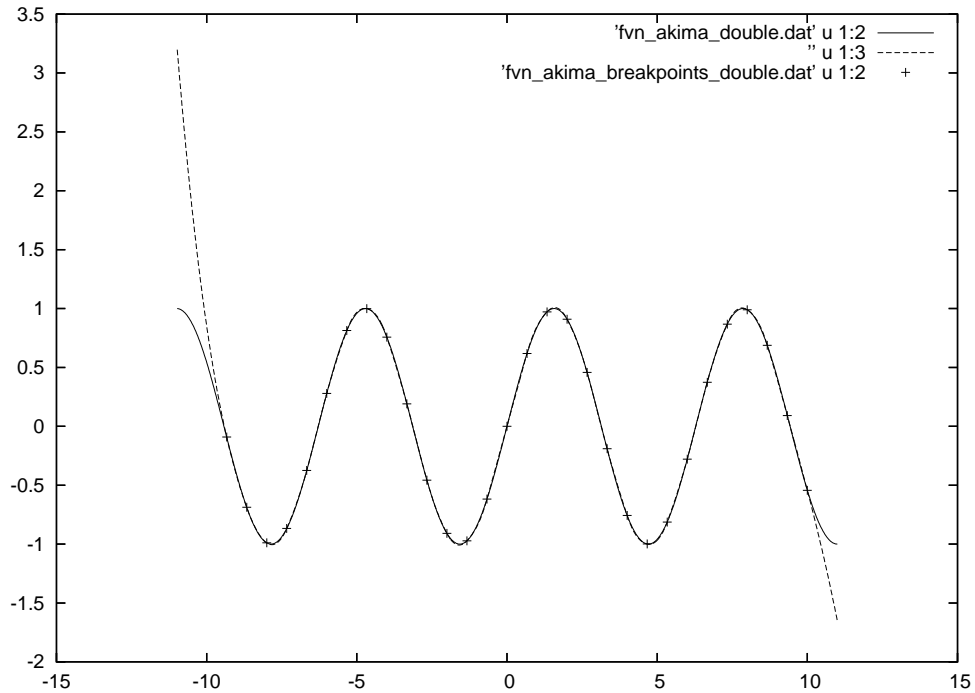


Figure 1: Akima Spline Interpolation

5 Least square polynomial

fvn provide a function to find a least square polynomial of a given degree, for real in single or double precision. It is performed using Lapack subroutine sgels (dgels), which solve this problem.

```
call fvn_lspoly(np,x,y,deg,coeff,status)
```

- np (in) is an integer equal to the number of points
- x(np) (in),y(np) (in) are the known coordinates
- deg (in) is an integer equal to the degree of the desired polynomial, it must be lower than np.
- coeff(deg+1) (out) on output contains the polynomial coefficients
- status (out) is an integer containing 0 if a problem occurred.

Example

Here's a simple example : we've got 13 measurement points and we want to find the least square degree 3 polynomial for these points :

```
program lsp
use fvn
implicit none

integer,parameter :: npoints=13,deg=3
integer :: status,i
```

```

real(kind=8) :: xm(npoints),ym(npoints),xstep,xc,yc
real(kind=8) :: coeff(deg+1)

xm = (/ -3.8,-2.7,-2.2,-1.9,-1.1,-0.7,0.5,1.7,2.,2.8,3.2,3.8,4. /)
ym = (/ -3.1,-2.,-0.9,0.8,1.8,0.4,2.1,1.8,3.2,2.8,3.9,5.2,7.5 /)

open(2,file='fvn_lsp_double_mesure.dat')
open(3,file='fvn_lsp_double_poly.dat')

do i=1,npoints
  write(2,44) xm(i),ym(i)
end do
close(2)

call fvn_d_lspoly(npoints,xm,ym,deg,coeff,status)

xstep=(xm(npoints)-xm(1))/1000.
do i=1,1000
  xc=xm(1)+(i-1)*xstep
  yc=poly(xc,coeff)
  write(3,44) xc,yc
end do
close(3)

44      FORMAT(4(1X,1PE22.14))

contains
function poly(x,coeff)
  implicit none
  real(8) :: x
  real(8) :: coeff(deg+1)
  real(8) :: poly
  integer :: i

  poly=0.

  do i=1,deg+1
    poly=poly+coeff(i)*x**(i-1)
  end do

end function
end program

```

The results are plotted on figure 2 .

6 Zero finding

fvn provide a routine for finding zeros of a complex function using Muller algorithm (only for double complex type). It is based on a version provided on the web by Hans D Mittelmann http://plato.asu.edu/ftp/other_software/muller.f.

```
call fvn_muller(f,eps,eps1,kn,nguess,n,x,itmax,infer,ier)
```

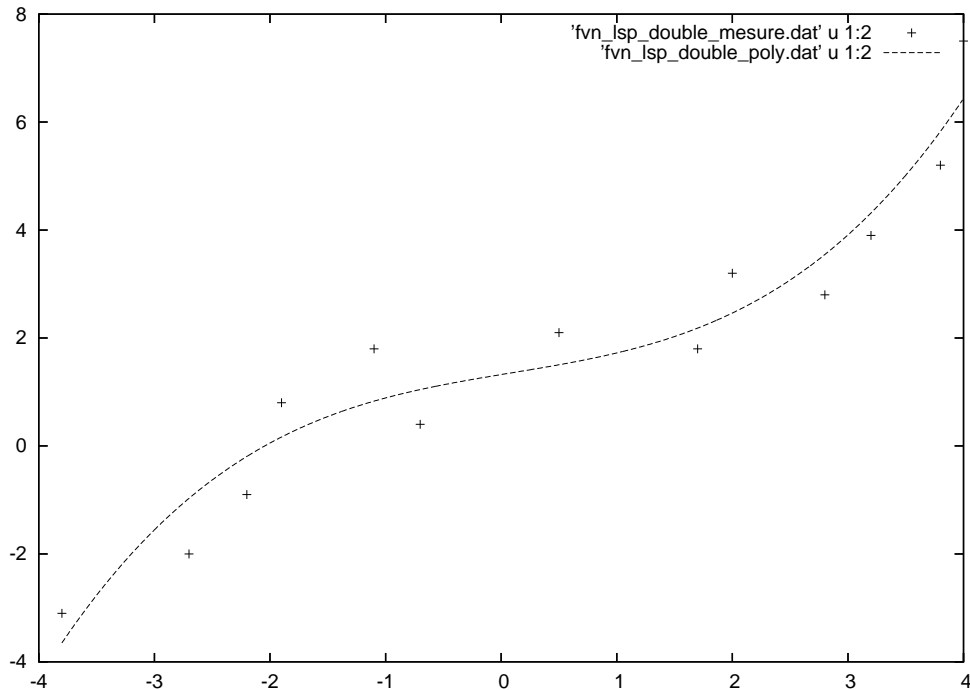


Figure 2: Least Square Polynomial

- `f` (in) is the complex function (kind=8) for which we search zeros
- `eps` (in) is a real(8) corresponding to the first stopping criterion : let $fp(z)=f(z)/p$ where $p = (z-z(1))*(z-z(2))*\dots*(z-z(k-1))$ and $z(1),\dots,z(k-1)$ are previously found roots. if $((cdabs(f(z))).le.eps)$.and. $(cdabs(fp(z))).le.eps$), then z is accepted as a root.
- `eps1` (in) is a real(8) corresponding to the second stopping criterion : a root is accepted if two successive approximations to a given root agree within `eps1`. Note that if either or both of the stopping criteria are fulfilled, the root is accepted.
- `kn` (in) is an integer equal to the number of known roots, which must be stored in $x(1),\dots,x(kn)$, prior to entry in the subroutine.
- `nguess` (in) is the number of initial guesses provided. These guesses must be stored in $x(kn+1),\dots,x(kn+nguess)$. `nguess` must be set equal to zero if no guesses are provided.
- `n` (in) is an integer equal to the number of new roots to be found.
- `x` (inout) is a complex(8) vector of length $kn+n$. $x(1),\dots,x(kn)$ on input must contain any known roots. $x(kn+1),\dots,x(kn+n)$ on input may, on user option, contain initial guesses for the n new roots which are to be computed. If the user does not provide an initial guess, zero is used. On output, $x(kn+1),\dots,x(kn+n)$ contain the approximate roots found by the subroutine.
- `itmax` (in) is an integer equal to the maximum allowable number of iterations per root.
- `infer` (out) is an integer vector of size $kn+n$. On output `infer(j)` contains the number of iterations used in finding the j -th root when convergence was achieved. If convergence was not obtained in `itmax` iterations, `infer(j)` will be greater than `itmax`

- `ier` (out) is an integer used as an error parameter. `ier = 33` indicates failure to converge within `itmax` iterations for at least one of the (`n`) new roots.

This subroutine always returns the last approximation for root `j` in `x(j)`. if the convergence criterion is satisfied, then `infer(j)` is less than or equal to `itmax`. if the convergence criterion is not satisfied, then `infer(j)` is set to either `itmax+1` or `itmax+k`, with `k` greater than 1. `infer(j) = itmax+1` indicates that muller did not obtain convergence in the allowed number of iterations. in this case, the user may wish to set `itmax` to a larger value. `infer(j) = itmax+k` means that convergence was obtained (on iteration `k`) for the deflated function $f_p(z) = f(z)/((z-z(1))\dots(z-z(j-1)))$ but failed for $f(z)$. in this case, better initial guesses might help or, it might be necessary to relax the convergence criterion.

Example

Example to find the ten roots of $x^{10} - 1$

```

program muller
  use fvn
  implicit none

  integer :: i,info
  complex(8),dimension(10) :: roots
  integer,dimension(10) :: infer
  complex(8), external :: f

  call fvn_z_muller(f,1.d-12,1.d-10,0,0,10,roots,200,infer,info)

  write(*,*) "Error code :",info
  do i=1,10
    write(*,*) roots(i),infer(i)
  enddo
end program

function f(x)
  complex(8) :: x,f
  f=x**10-1
end function

```

7 Numerical integration

Using an integrated slightly modified version of quadpack <http://www.netlib.org/quadpack>, `fvn` provide adaptative numerical integration (Gauss Kronrod) of real functions of 1 and 2 variables. `fvn` also provide a function to calculate Gauss-Legendre abscissas and weight, and a simple non adaptative integration subroutine. All routines exists only in `fvn` for double precision real.

7.1 Gauss Legendre Abscissas and Weigth

This subroutine was inspired by Numerical Recipes routine `gauleg`.

```
call fvn_gauss_legendre(n,qx,qw)
```

- `n` (in) is an integer equal to the number of Gauss Legendre points
- `qx` (out) is a `real(8)` vector of length `n` containing the abscissas.

- `qw` (out) is a real(8) vector of length `n` containing the weights.

This subroutine computes `n` Gauss-Legendre abscissas and weights

7.2 Gauss Legendre Numerical Integration

`call fvn_gl_integ(f,a,b,n,res)`

- `f` (in) is a real(8) function to integrate
- `a` (in) and `b` (in) are real(8) respectively lower and higher bound of integration
- `n` (in) is an integer equal to the number of Gauss Legendre points to use
- `res` (out) is a real(8) containing the result

This function is a simple Gauss Legendre integration subroutine, which evaluate the integral of function `f` as in equation 3 using `n` Gauss-Legendre pairs.

7.3 Gauss Kronrod Adaptative Integration

This kind of numerical integration is an iterative procedure which try to achieve a given precision.

7.3.1 Numerical integration of a one variable function

`call fvn_integ_1_gk(f,a,b,epsabs,epsrel,key,res,abserr,ier,limit)`

This routine evaluate the integral of function `f` as in equation 3

- `f` (in) is an external real(8) function of one variable
- `a` (in) and `b` (in) are real(8) respectively lower and higher bound of integration
- `epsabs` (in) and `epsrel` (in) are real(8) respectively desired absolute and relative error
- `key` (in) is an integer between 1 and 6 correspondind to the Gauss-Kronrod rule to use :
 - 1 : 7 - 15 points
 - 2 : 10 - 21 points
 - 3 : 15 - 31 points
 - 4 : 20 - 41 points
 - 5 : 25 - 51 points
 - 6 : 30 - 61 points
- `res` (out) is a real(8) containing the estimation of the integration.
- `abserr` (out) is a real(8) equal to the estimated absolute error
- `ier` (out) is an integer used as an error flag
 - 0 : no error
 - 1 : maximum number of subdivisions allowed has been achieved. one can allow more subdivisions by increasing the value of `limit` (and taking the according dimension adjustments into account). however, if this yield no improvement it is advised to analyze the integrand in order to determine the integration difficulties. If the position of a local difficulty can be determined (i.e.singularity, discontinuity within the interval) one will probably gain from splitting up the interval at this point and calling the integrator on the subranges. If possible, an appropriate special-purpose integrator should be used which is designed for handling the type of difficulty involved.

- 2 : the occurrence of roundoff error is detected, which prevents the requested tolerance from being achieved.
 - 3 : extremely bad integrand behaviour occurs at some points of the integration interval.
 - 6 : the input is invalid, because (epsabs.le.0 and epsrel.lt.max(50*rel.mach.acc.,0.5d-28)) or limit.lt.1 or lenw.lt.limit*4. result, abserr, neval, last are set to zero. Except when lenw is invalid, iwork(1), work(limit*2+1) and work(limit*3+1) are set to zero, work(1) is set to a and work(limit+1) to b.
- limit (in) is an optional integer equal to maximum number of subintervals in the partition of the given integration interval (a,b). If the parameter is not present a default value of 500 will be used.

$$\int_a^b f(x) dx \quad (3)$$

7.3.2 Numerical integration of a two variable function

call fvn_integ_2_gk(f,a,b,g,h,epsabs,epsrel,key,res,abserr,ier,limit)

This function evaluate the integral of a function $f(x,y)$ as defined in equation 4. The parameters of same name as in the previous paragraph have exactly the same function and behaviour thus only what differs is described here

- a (in) and b (in) are real(8) corresponding respectively to lower and higher bound of integration for the x variable.
- g(x) (in) and h(x) (in) are external functions describing the lower and higher bound of integration for the y variable as a function of x.

$$\int_a^b \int_{g(x)}^{h(x)} f(x,y) dy dx \quad (4)$$

Example

```

program integ
  use fvn
  implicit none

  real(8), external :: f1,f2,g,h
  real(8) :: a,b,epsabs,epsrel,abserr,res
  integer :: key,ier

  a=0.
  b=1.
  epsabs=1d-8
  epsrel=1d-8
  key=2
  call fvn_d_integ_1_gk(f1,a,b,epsabs,epsrel,key,res,abserr,ier,500)
  write(*,*) "Integration of x*x between 0 and 1 : "
  write(*,*) res

  call fvn_d_integ_2_gk(f2,a,b,g,h,epsabs,epsrel,key,res,abserr,ier,500)
  write(*,*) "Integration of x*y between 0 and 1 on both x and y : "
  write(*,*) res

```

```

end program

function f1(x)
  implicit none
  real(8) :: x,f1
  f1=x*x
end function

function f2(x,y)
  implicit none
  real(8) :: x,y,f2
  f2=x*y
end function

function g(x)
  implicit none
  real(8) :: x,g
  g=0.
end function

function h(x)
  implicit none
  real(8) :: x,h
  h=1.
end function

```

8 Special functions

Special functions are available in fvn by using an implementation of fnlib <http://www.netlib.org/fn> with some additions. This can be used separately from the rest of fvn by using the module `fvn_fnlib` and linking the library `libfvn_fnlib.a`. The module provides a generic interfaces to all the routines. Specific names of the routines are given in the description.

Important Note Due to the addition of fnlib to fvn, some functions that were in fvn and are redundant will be removed from fvn, so update your code now and replace them with the fnlib version. These are listed here after :

- `fvn_z_acos` replaced by `acos`
- `fvn_z_asin` replaced by `asin`
- `fvn_d_asinh` replaced by `asinh`
- `fvn_d_acosh` replaced by `acosh`
- `fvn_s_csevl` replaced by `csevl`
- `fvn_d_csevl` replaced by `csevl`
- `fvn_d_factorial` replaced by `fac`
- `fvn_d_lngamma` replaced by `alngam`

8.1 Elementary functions

8.1.1 carg

`carg(z)`

- `z` (in) is a complex

This function evaluates the argument of the complex `z`. That is θ for $z = \rho e^{i\theta}$.

Specific interfaces : `carg,zarg`

8.1.2 cbrt

`cbrt(x)`

- `x` is a real or complex

This function evaluates the cubic root of the argument `x`.

Specific interfaces : `cbrt,dcbrt,ccbrt,zcbrt`

8.1.3 expri

`expri(x)`

- `x` is a real or complex

This function evaluates $\frac{e^x-1}{x}$.

Specific interfaces : `expri,dexpri,cexpri,zexpri`

8.1.4 log10

`log10(x)`

- `x` is a real or complex

This function is an extension of the intrinsic function `log10` to complex arguments.

Specific interfaces : `clog10,zlog10`

8.1.5 alnrel

`alnrel(x)`

- `x` is a real or complex

This function evaluates $\ln(1+x)$.

Specific interfaces : `alnrel,dlnrel,clnrel,zlnrel`

8.2 Trigonometry

8.2.1 tan

`tan(x)`

- `x` is a real or complex

This function evaluates the tangent of the argument. It is an extension of the intrinsic function `tan` to complex arguments.

Specific interfaces : `ctan,ztan`

8.2.2 cot

`cot(x)`

- x is a real or complex

This function evaluate the cotangent of the argument.

Specific interfaces : `cot,dcot,ccot,zcot`

8.2.3 sindg

`sindg(x)`

- x is a real

This function evaluate the sinus of the argument expressed in degrees.

Specific interfaces : `sindg,dsindg`

8.2.4 cosdg

`cosdg(x)`

- x is a real

This function evaluate the cosinus of the argument expressed in degrees.

Specific interfaces : `cosdg,dcosdg`

8.2.5 asin

`asin(x)`

- x is a real or complex

This function evaluates the arc sine of the argument. It is an extension of the intrinsic function `asin` to complex arguments.

Specific interfaces : `casin,zasin`

8.2.6 acos

`acos(x)`

- x is a real or complex

This function evaluates the arc cosine of the argument. It is an extension of the intrinsic function `acos` to complex arguments.

Specific interfaces : `caacos,zacos`

8.2.7 atan

`atan(x)`

- x is a real or complex

This function evaluates the arc tangent of the argument. It is an extension of the intrinsic function `atan` to complex arguments.

Specific interfaces : `catan,zatan`

8.2.8 atan2

`atan2(x,y)`

- x,y are real or complex

This function evaluates the arc tangent of $\frac{x}{y}$. It is an extension of the intrinsic function `atan2` to complex arguments.

Specific interfaces : `catan2,zatan2`

8.2.9 sinh

`sinh(x)`

- x is a real or complex

This function evaluates the hyperbolic sine of the argument. It is an extension of the intrinsic function `sinh` to complex arguments.

Specific interfaces : `csinh,zsinh`

8.2.10 cosh

`cosh(x)`

- x is a real or complex

This function evaluates the hyperbolic cosine of the argument. It is an extension of the intrinsic function `cosh` to complex arguments.

Specific interfaces : `ccosh,zcosh`

8.2.11 tanh

`tanh(x)`

This function evaluates the hyperbolic tangent of the argument. It is an extension of the intrinsic function `tanh` to complex arguments.

Specific interfaces : `ctanh,ztanh`

8.2.12 asinh

`asinh(x)`

- x is a real or complex

This function evaluates the arc hyperbolic sine of the argument.

Specific interfaces : `asinh,dasinh,casinh,zasinh`

8.2.13 acosh

`acosh(x)`

- x is a real or complex

This function evaluates the arc hyperbolic cosine of the argument.

Specific interfaces : `acosh,dacosh,cacosh,zacosh`

8.2.14 atanh

atanh(x)

- x is a real or complex

This function evaluates the arc hyperbolic tangent of the argument.

Specific interfaces : `atanh`, `datanh`, `catanh`, `zatanh`

8.3 Exponential Integral and related

8.3.1 ei

ei(x)

- x is a real

This function evaluates the exponential integral for argument greater than 0 and the Cauchy principal value for argument less than 0. It is define by equation 5 for $x \neq 0$.

$$ei(x) = - \int_{-x}^{\infty} \frac{e^{-t}}{t} dt \quad (5)$$

Specific interfaces : `ei`, `dei`

8.3.2 e1

e1(x)

- x is a real

This function evaluates the exponential integral for argument greater than 0 and the Cauchy principal value for argument less than 0. It is define by equation 6 for $x \neq 0$.

$$e1(x) = \int_x^{\infty} \frac{e^{-t}}{t} dt \quad (6)$$

Specific interfaces : `e1`, `de1`

8.3.3 ali

ali(x)

- x is a real

This function evaluates the logarithm integral. it is define by equation 7 for $x > 0$ and $x \neq 1$.

$$ali(x) = - \int_0^x \frac{dt}{\ln(x)} \quad (7)$$

Specific interfaces : `ali`, `dli`

8.3.4 si

si(x)

- x is a real

This function evaluates the sine integral defined by equation 8.

$$si(x) = \int_0^x \frac{\sin(t)}{t} dt \quad (8)$$

Specific interfaces : `si`, `dsi`

8.3.5 ci

`ci(x)`

- x is a real

This function evaluates the cosine integral defined by equation 9 where $\gamma \approx 0.57721566$ represent Euler's constant.

$$ci(x) = \gamma + \ln(x) + \int_0^x \frac{1 - \cos(t)}{t} dt \quad (9)$$

Specific interfaces : `ci, dci`

8.3.6 cin

`cin(x)`

- x is a real

This function evaluates the cosine integral alternate definition given by equation 10.

$$cin(x) = \int_0^x \frac{1 - \cos(t)}{t} dt \quad (10)$$

Specific interface : `cin, dcin`

8.3.7 shi

$$shi(x) \quad (11)$$

- x is a real

This function evaluates the hyperbolic sine integral defined by equation 12.

$$shi(x) = \int_0^x \frac{\sinh(t)}{t} dt \quad (12)$$

Specific interfaces : `shi, dshi`

8.3.8 chi

`chi(x)`

- x is a real

This function evaluates the hyperbolic cosine integral defined by equation 13 where $\gamma \approx 0.57721566$ represent Euler's constant.

$$chi(x) = \gamma + \ln(x) + \int_0^x \frac{\cosh(t) - 1}{t} dt \quad (13)$$

Specific interfaces : `chi, dchi`

8.3.9 cinh

`cinh(x)`

- x is a real

This function evaluates the hyperbolic cosine integral alternate definition given by equation 14.

$$cinh(x) = \int_0^x \frac{\cosh(t) - 1}{t} dt \quad (14)$$

Specific interfaces : `cinh, dcinh`

8.4 Gamma function and related

8.4.1 fac

fac(n)
dfac(n)

- n is an integer

This function return $n!$ as a real(4) or real(8) for dfac. There's no generic interface for this one.

Specific interfaces : `fac,dfac`

8.4.2 binom

binom(n,m)
dbinom(n,m)

- n,m are integers

This function return the binomial coefficient defined by equation 15 with $n \geq m \geq 0$. binom returns a real(4), dbinom a real(8). There's no generic interface for this one.

$$\text{binom}(n, m) = C_n^m = \frac{n!}{m!(n-m)!} \quad (15)$$

Specific interfaces : `binom,dbinom`

8.4.3 gamma

gamma(x)

- x is a real or complex

This function evaluates $\Gamma(x)$ defined by equation 16.

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt \quad (16)$$

Note that $n! = \Gamma(n+1)$.

Specific interfaces : `gamma,dgamma,cgamma,zgamm`

8.4.4 gamr

gamr(x)

- x is a real or complex

This function evaluates the reciprocal gamma function $\text{gamr}(x) = \frac{1}{\Gamma(x)}$

8.4.5 alngam

alngam(x)

- x is a real or complex

This function evaluates $\ln(|\Gamma(x)|)$

Specific interfaces : `alngam,dlngam,clngam,zlngam`

8.4.6 algams

call `algams(x,algam,sgngam)`

- `x` (in) is a real
- `algam` (out) is a real
- `sgngam` (out) is a real

This subroutine evaluates the logarithm of the absolute value of gamma and the sign of gamma. $algam = \ln(|\Gamma(x)|)$ and $sgngam = 1.0$ or -1.0 according to the sign of $\Gamma(x)$.

Specific interfaces : `algams,dlgams`

8.4.7 gami

`gami(a,x)`

- `x` is a positive real
- `a` is a strictly positive real

This function evaluates the incomplete gamma function defined by equation 17.

$$gami(a,x) = \gamma(a,x) = \int_0^x t^{a-1} e^{-t} dt \quad (17)$$

Specific interfaces : `gami,dgami`

8.4.8 gamic

`gamic(a,x)`

- `x` is a positive real
- `a` is a real

This function evaluates the complementary incomplete gamma function defined by equation 18.

$$gamic(a,x) = \Gamma(a,x) = \int_x^\infty t^{a-1} e^{-t} dt \quad (18)$$

Specific interfaces : `gamic,dgamic`

8.4.9 gamit

`gamit(a,x)`

- `x` is a positive real
- `a` is a real

This function evaluates the Tricomi's incomplete gamma function defined by equation 19.

$$gamit(a,x) = \gamma^*(a,x) = \frac{x^{-a} \gamma(a,x)}{\Gamma(a)} \quad (19)$$

Specific interfaces : `gamit,dgamit`

8.4.10 psi

psi(x)

- x is a real or complex

This function evaluates the psi function which is the logarithm derivative of the gamma function as defined in equation 20.

$$psi(x) = \psi(x) = \frac{d}{dx} \ln(\Gamma(x)) \quad (20)$$

x must not be zero or a negative integer.

Specific interfaces : psi,dpsi,cpsi,zpsi

8.4.11 poch

poch(a,x)

- x is a real
- a is a real

This function evaluates a generalization of Pochhammer's symbol.

Pochhammer's symbol for n a positive integer is given by equation 21

$$(a)_n = a(a-1)(a-2)\dots(a-n+1) \quad (21)$$

The generalization of Pochhammer's symbol is given by equation 22

$$poch(a,x) = (a)_x = \frac{\Gamma(a+x)}{\Gamma(a)} \quad (22)$$

Specific interfaces : poch,dpoch

8.4.12 poch1

poch1(a,x)

- x is a real
- a is a real

This function is defined by equation 23. It is useful for certain situations, especially when x is small.

$$poch1(a,x) = \frac{(a)_x - 1}{x} \quad (23)$$

Specific interfaces : poch1,dpoch1

8.4.13 beta

beta(a,b)

- a,b are real positive or complex

This function evaluates β function defined by equation 24.

$$beta(a,b) = \beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \quad (24)$$

Specific interfaces : beta,dbeta,cbeta,zbeta

8.4.14 `albeta`

`albeta(a,b)`

- a,b are real positive or complex

This function evaluates the natural logarithm of beta function : $\ln(\beta(a,b))$

Specific interfaces : `albeta,dlbeta,clbeta,zlbeta`

8.4.15 `betai`

`betai(x,pin,qin)`

- x is a real in [0,1]
- pin and qin are strictly positive real

This function evaluates the incomplete beta function ratio, that is the probability that a random variable from a beta distribution having parameters pin and qin will be less than or equal to x. It is defined by equation 25.

$$betai(x,pin,qin) = I_x(pin,qin) = \frac{1}{\beta(pin,qin)} \int_0^x t^{pin-1}(1-t)^{qin-1} dt \quad (25)$$

Specific interfaces : `betai,dbetai`

8.5 Error function and related

8.5.1 `erf`

`erf(x)`

- x is a real

This function evaluates the error function defined by equation 26.

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \quad (26)$$

Specific interfaces : `erf,derf`

8.5.2 `erfc`

`erfc(x)`

- x is a real

This function evaluates the complementary error function defined by equation 27.

$$erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt \quad (27)$$

Specific interfaces : `erfc,derfc`

8.5.3 `daws`

`daws(x)`

- x is a real

This function evaluates Dawson's function defined by equation 28.

$$daws(x) = e^{-x^2} \int_0^x e^{t^2} dt \quad (28)$$

Specific interfaces : `daws,ddaws`

8.6 Bessel functions and related

8.6.1 `bsj0`

`bsj0(x)`

- `x` is a real

This function evaluates Bessel function of the first kind of order 0 defined by equation 29.

$$bsj0(x) = J_0(x) = \frac{1}{\pi} \int_0^\pi \cos(x \sin(\theta)) d\theta \quad (29)$$

Specific interfaces : `besj0,dbesj0`

8.6.2 `bsj1`

`bsj1(x)`

- `x` is a real

This function evaluates Bessel function of the first kind of order 1 defined by equation 30.

$$bsj1(x) = J_1(x) = \frac{1}{\pi} \int_0^\pi \cos(x \sin(\theta) - \theta) d\theta \quad (30)$$

Specific interfaces : `besj1,dbesj1`

8.6.3 `bsjn`

`bsjn(n,x,factor,big)`

- `n` is an integer
- `x` is a real
- `factor` is an optional integer
- `big` is an optional real

This function evaluates Bessel function of the first kind of order `n` (plotted in figure 3). These functions satisfy the recurrent relation 31.

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x) \quad (31)$$

This relation is directly used in upward direction to compute $J_n(x)$ for $x > n$. However it is unstable for $x < n$, therefore a Miller's Algorithm is used. The principle of this method is to use the recurrent relation downward from an arbitrary higher than `n` order with an arbitrary seed and then normalize the solution with 32

$$1 = J_0 + 2J_2 + 2J_4 + 2J_6 + \dots \quad (32)$$

The optional parameters `factor` and `big` can be used to modify the behaviour of the algorithm. `factor` is used in determining the arbitrary starting order (an even integer near $n + \sqrt{\text{factor } n}$), the default `factor` value is 40. `big` is a real determining the threshold for which anti-overflow counter measures has to be taken, default value is 1.10^{10}

By convenience, the routine accept $n = 0$ and $n = 1$, in that cases a call to `bsj0(x)` or `bsj1(x)` is actually performed.

Specific interfaces : `besjn,dbesjn`

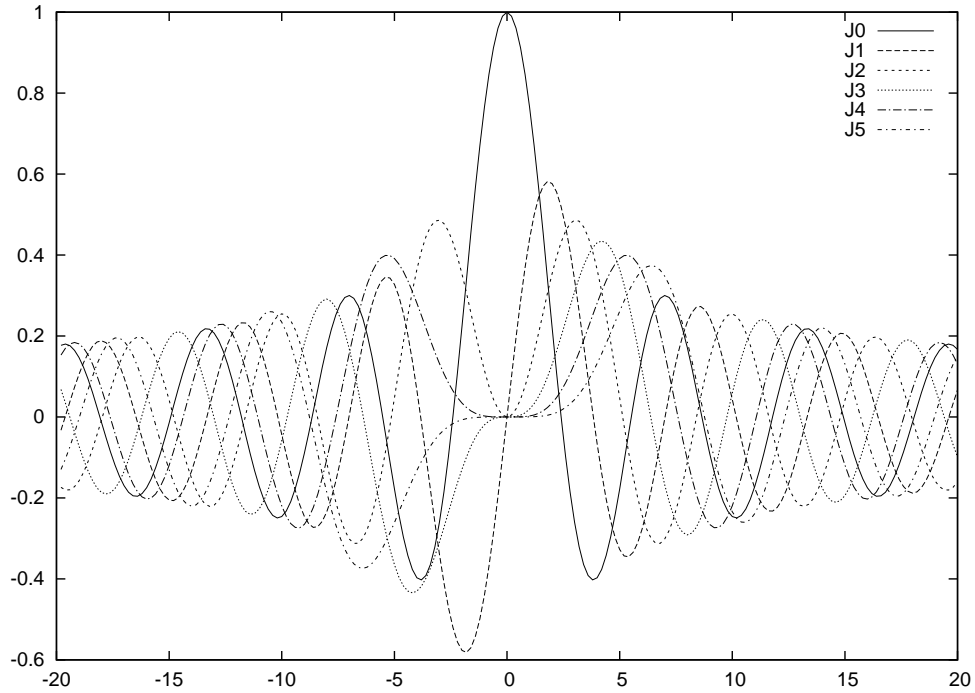


Figure 3: Bessel J_n functions family

8.6.4 bsy0

`bsy0(x)`

- x is a strictly positive real

This function evaluates the Bessel function of the second kind of order 0 defined by equation 33

$$bsy0(x) = Y_0(x) = \frac{1}{\pi} \int_0^\pi \sin(x \sin(\theta)) d\theta - \frac{2}{\pi} \int_0^\infty e^{-x \sinh(t)} dt \quad (33)$$

Specific interfaces : `bsy0`, `dbesy0`

8.6.5 bsy1

`bsy1(x)`

- x is a strictly positive real

This function evaluates the Bessel function of the second kind of order 1 defined by equation 34.

$$bsy1(x) = Y_1(x) = -\frac{1}{\pi} \int_0^\pi \sin(\theta - x \sin(\theta)) d\theta - \frac{1}{\pi} \int_0^\infty (e^t - e^{-t}) e^{-x \sinh(t)} dt \quad (34)$$

Specific interfaces : `bsy1`, `dbesy1`

8.6.6 bsyn

`bsyn(n, x)`

- n is an integer

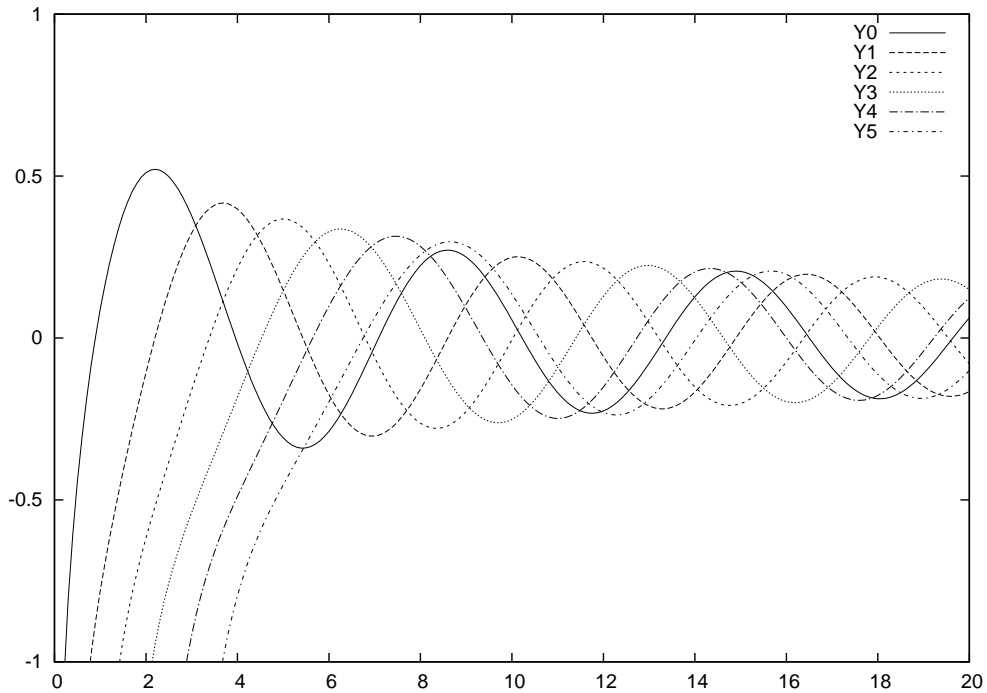


Figure 4: Bessel Y_n functions family

- x is a strictly positive real

This function evaluates the Bessel function of the second kind of order n (plotted in figure 4). These functions satisfy the recurrent relation 35.

$$Y_{n+1}(x) = \frac{2n}{x}Y_n(x) - Y_{n-1}(x) \quad (35)$$

This recurrent relation is directly used in the upward direction to compute $Y_n(x)$.

By convenience, the routine accept $n = 0$ and $n = 1$, in that cases a call to $bsy0(x)$ or $bsy1(x)$ is actually performed.

Specific interfaces : `besyn,dbesyn`

8.6.7 bsi0

`bsi0(x)`

- x is a real

This function evaluates the Bessel function of the third kind of order 0 defined by equation 36.

$$bsi0(x) = I_0(x) = \frac{1}{\pi} \int_0^\pi \cosh(x \cos(\theta)) d\theta \quad (36)$$

Specific interfaces : `besi0,dbesi0`

8.6.8 bsi1

`bsi1(x)`

- x is a real

This function evaluates the Bessel function of the third kind of order 1 defined by equation 37.

$$bsi1(x) = I_1(x) = \frac{1}{\pi} \int_0^\pi e^{x \cos(\theta)} \cos(\theta) d\theta \quad (37)$$

Specific interfaces : `bsi1,dbesi1`

8.6.9 bsin

`bsin(n,x,factor,big)`

- n is an integer
- x is a real
- `factor` is an optional integer
- `big` is an optional real

This function evaluates Bessel function of the third kind of order n (plotted in figure 5). These functions satisfy the recurrent relation 38

$$I_{n+1}(x) = -\frac{2n}{x} I_n(x) + I_{n-1}(x) \quad (38)$$

This relation is unstable in the upward direction, therefore a Miller's Algorithm is used to evaluate the function. Even if there's a usable normalization relation 39, it is not used in the routine, instead normalization is done by a simple call to `bsi0(x)`.

$$1 = I_0 - 2I_2 + 2I_4 - 2I_6 + \dots \quad (39)$$

The optional parameters `factor` and `big` can be used to modify the behaviour of the algorithm. `factor` is used in determining the arbitrary starting order (an even integer near $n + \sqrt{\text{factor } n}$), the default `factor` value is 40. `big` is a real determining the threshold for which anti-overflow counter measures has to be taken, default value is 1.10^{10}

By convenience, the routine accept $n = 0$ and $n = 1$, in that cases a call to `bsi0(x)` or `bsi1(x)` is actually performed.

Specific interfaces : `bsin,dbesin`

8.6.10 bsk0

`bsk0(x)`

- x is a strictly positive real

This function evaluates the modified Bessel function of the second kind of order 0 defined by equation 40

$$bsk0(x) = K_0(x) = \int_0^\infty \cos(x \sinh(t)) dt \quad (40)$$

Specific interfaces : `bsk0,dbesk0`

8.6.11 bsk1

`bsk1(x)`

- x is a strictly positive real

This function evaluates the modified Bessel function of the second kind of order 1 defined by equation 41

$$bsk1(x) = K_1(x) = \int_0^\infty \sin(x \sinh(t)) \sinh(t) dt \quad (41)$$

Specific interfaces : `bsk1,dbesk1`

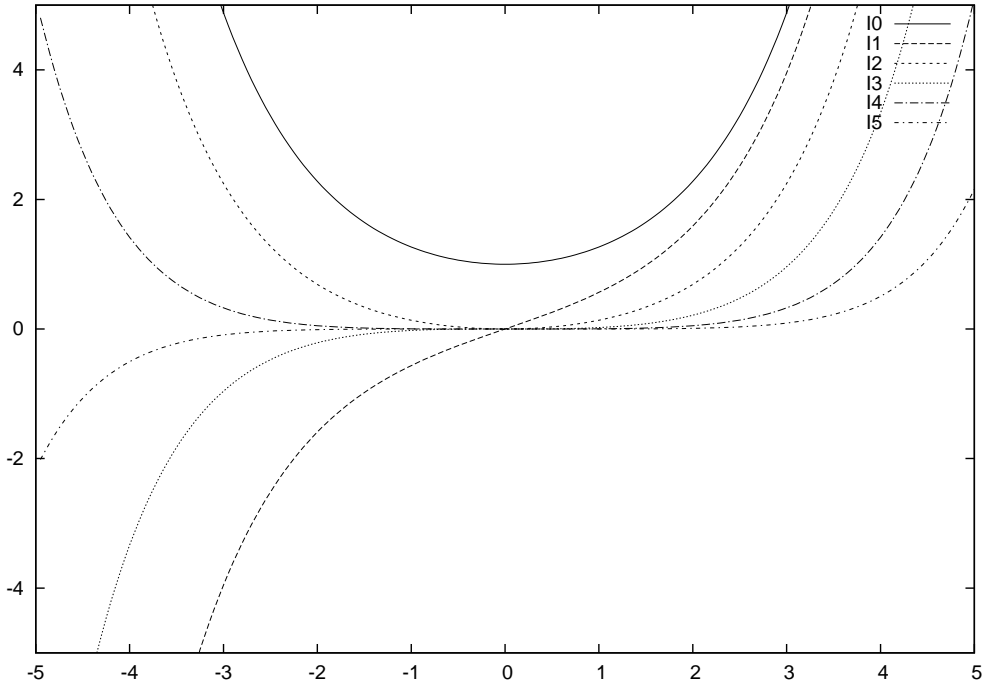


Figure 5: Bessel I_n functions family

8.6.12 bskn

`bskn(n, x)`

- n is an integer
- x is strictly positive real

This function evaluates the modified Bessel function of the second kind of order n (plotted in figure 6). These functions satisfy the recurrent relation 42

$$K_{n+1}(x) = \frac{2n}{x}K_n(x) + K_{n-1}(x) \quad (42)$$

This recurrent relation is directly used in the upward direction to compute $K_n(x)$.

By convenience, the routine accept $n = 0$ and $n = 1$, in that cases a call to `bsk0(x)` or `bsk1(x)` is actually performed.

Specific interface : `beskn, dbeskn`

8.6.13 bsi0e

`bsi0e(x)`

- x is a real

This function evaluates $e^{-|x|}I_0(x)$

Specific interfaces : `bsi0e, dbsi0e`

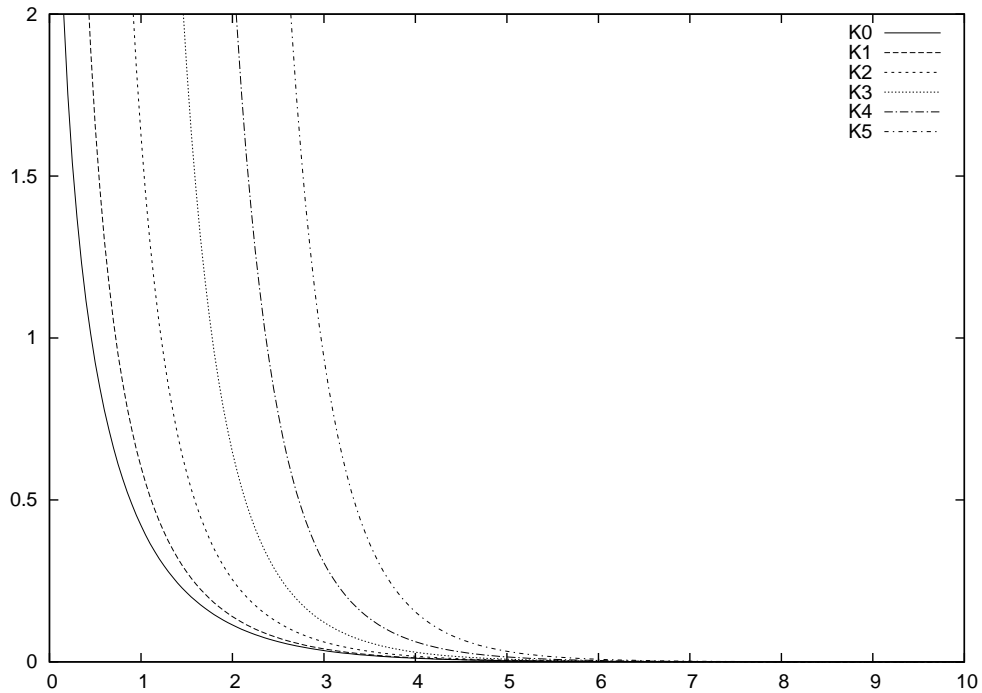


Figure 6: Bessel K_n functions family

8.6.14 `bsile`

`bsile(x)`

- x is a real

This function evaluates $e^{-|x|}I_1(x)$

Specific interfaces : `bsile`, `dbsile`

8.6.15 `bsk0e`

`bsk0e(x)`

- x is a strictly positive real

This function evaluates $e^x K_0(x)$

Specific interfaces : `bsk0e`, `dbsk0e`

8.6.16 `bsk1e`

`bsk1e(x)`

- x is a strictly positive real

This function evaluates $e^x K_1(x)$

Specific interfaces : `bsk1e`, `dbsk1e`

8.6.17 bsk

call bsk(xnu,x,nin,bk)

- xnu (in) is a real with $|xnu| < 1$. It's the fractional order
- x (in) is a real. The value for which the sequence of Bessel functions is to be evaluated.
- nin (in) is an integer.
- bk (out) is a real vector of length abs(nin), containing the values of the function.

This subroutine evaluates a sequence of modified Bessel function of the second kind of fractional order.

If nin is positive, on completion $bk(1) = K_\nu(x), bk(2) = K_{\nu+1}(x), \dots, bk(nin) = K_{\nu+nin-1}(x)$. If nin is negative, on completion $bk(1) = K_\nu(x), bk(2) = K_{\nu-1}(x), \dots, bk(|nin|) = K_{\nu+nin+1}(x)$.

Specific interfaces : **bsks, dbsks**

8.6.18 bsks

call bsks(xnu,x,nin,bke)

- xnu (in) is a real with $|xnu| < 1$. It's the fractional order
- x (in) is a real. The value for which the sequence of exponentially scaled Bessel functions is to be evaluated.
- nin (in) is an integer. Number of elements in the sequence.
- bke (out) is a real vector of length abs(nin), containing the values of the function.

This subroutine evaluates a sequence of exponentially scaled modified Bessel function of the second kind of fractional order.

If nin is positive, on completion $bk(1) = e^x K_\nu(x), bk(2) = e^x K_{\nu+1}(x), \dots, bk(nin) = e^x K_{\nu+nin-1}(x)$. If nin is negative, on completion $bk(1) = e^x K_\nu(x), bk(2) = e^x K_{\nu-1}(x), \dots, bk(|nin|) = e^x K_{\nu+nin+1}(x)$.

Specific interfaces : **bskes, dbskes**

8.7 Airy function and related

8.7.1 ai

ai(x)

- x is a real

This function evaluates the airy function defined by equation 43

$$Ai(x) = \frac{1}{\pi} \int_0^\infty \cos\left(xt + \frac{1}{3}t^3\right) dt \quad (43)$$

Specific interfaces : **ai, dai**

8.7.2 bi

bi(x)

- x is a real

This function evaluates the Airy function of the second kind defined by equation 44

$$Bi(x) = \frac{1}{\pi} \int_0^\infty e^{xt - \frac{1}{3}t^3} dt + \frac{1}{\pi} \int_0^\infty \sin\left(xt + \frac{1}{3}t^3\right) dt \quad (44)$$

Specific interfaces : **bi, dbi**

8.7.3 aid

aid(x)

- x is a real

This function evaluates the derivative of the Airy function, $aid(x) = \frac{d}{dx}Ai(x)$.

Specific interface : **aid,daid**

8.7.4 bid

bid(x)

- x is a real

This function evaluates the derivative of the Airy function of the second kind, $bid(x) = \frac{d}{dx}Bi(x)$.

Specific interfaces : **bid,dbid**

8.7.5 aie

aie(x)

- x is a real

This function evaluates the exponentially scaled Airy function defined in equation 45.

$$aie(x) = \begin{cases} Ai(x) & \text{if } x \leq 0 \\ e^{\frac{2}{3}x^{\frac{3}{2}}} Ai(x) & \text{if } x > 0 \end{cases} \quad (45)$$

Specific interfaces : **aie,daie**

8.7.6 bie

bie(x)

- x is a real

This function evaluates the exponentially scaled Airy function of the second kind defined in equation 46.

$$bie(x) = \begin{cases} Bi(x) & \text{if } x \leq 0 \\ e^{-\frac{2}{3}x^{\frac{3}{2}}} Bi(x) & \text{if } x > 0 \end{cases} \quad (46)$$

Specific interfaces : **bie,dbie**

8.7.7 aide

aide(x)

- x is a real

This function evaluates the exponentially scaled derivative of the Airy function as defined in equation 47.

$$aide(x) = \begin{cases} Ai'(x) & \text{if } x \leq 0 \\ e^{\frac{2}{3}x^{\frac{3}{2}}} Ai'(x) & \text{if } x > 0 \end{cases} \quad (47)$$

Specific interfaces : **aide,daide**

8.7.8 `bide`

`bide(x)`

- `x` is a real

This function evaluates the exponentially scaled derivative of the Airy function of the second kind as defined in equation 48.

$$bie(x) = \begin{cases} Bi'(x) & \text{if } x \leq 0 \\ e^{-\frac{2}{3}x^{\frac{3}{2}}} Bi'(x) & \text{if } x > 0 \end{cases} \quad (48)$$

Specific interfaces : `bide,dbide`

8.8 Miscellaneous functions

8.8.1 `spenc`

`spenc(x)`

- `x` is a real

This function evaluates Spence function defined in equation 49.

$$spenc(x) = s(x) = - \int_0^x \frac{\ln(|1-t|)}{t} dt \quad (49)$$

Specific interfaces : `spenc,dspenc`

8.8.2 `inits`

`inits(os,nos,eta)`

- `os` is a real vector of length `nos`, containing the coefficients in an orthogonal series.
- `nos` is an integer
- `eta` is a real (Warning `eta` is a real(4) even with the double precision version) representing the requested accuracy.

This function initialize the orthogonal series so that `inits` is the number of terms needed to insure the error is no larger than `eta`.

Specific interfaces : `inits,initds`

8.8.3 `csevl`

`csevl(x,cs,n)`

- `x` is a real in `[-1,1]`
- `cs` is a real vector of length `n` containing the coefficients of the Chebyshev serie.
- `n` is an integer

This function evaluates the Chebyshev series whose coefficients are stored in `cs`.

Specific interfaces : `csevl,dcsevl`